Lectures on Strings, D-branes and Gauge Theories[*]

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Abstract

In these lectures we review the basic ideas of perturbative and non-perturbative string theory. On the non-perturbative side we give an introduction to D-branes and string duality. The elementary concepts of non-BPS branes and noncommutative gauge theories are also discussed.

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I. INTRODUCTION

String theory is by now, beyond the standard model of particle physics, the best and the most sensible understanding of all the matter and their interactions in an unified scheme. There are well known the 'esthetical' problems arising in the heart of the standard model of particles, such as: the abundance of free parameters, the origin of flavor and the gauge group, etc. It is also generally accepted that these problems require to be answered. Thus the standard model can be seen as the low energy effective theory of a more fundamental theory which can solve the mentioned problems. It is also clear that the quantum mechanics and general relativity cannot be reconciliated in the context of a perturbative quantum field theory of point particles. Hence the nonrenormalizability of the general relativity can be seen as a genuine evidence that it is just an effective field theory and new physics associated to the fast degrees of freedom should exist at higher energies. String theory propose that these fast degrees of freedom are precisely the strings at the perturbative level and at the non-perturbative level the relevant degrees of freedom are, in addition to the strings, higher-dimensional extended objects called D-branes (dual degrees of freedom). The study of theories involving D-branes is just in the starting stage and many surprises surely are coming up. Thus we are still at an exploratory stage of the whole structure of the string theory. Therefore the theory is far to be completed and we cannot give yet concrete physical predictions to take contact with collider experiments and/or astrophysical observations. However many aspects of theoretic character, necessary in order to make of string theory a physical theory, are quickly in progress. The purpose of these lectures are to overview the basic ideas to understand these progresses. This paper is an extended version of the lectures presented at the *Third Workshop on Gravitation and Mathematical Physics* at León Gto. México. We don't pretend to be exhaustive and we will limit ourselves to describe some basic elements of string theory and some particular new developments as: non-BPS branes and noncommutative gauge theories. We apologize for omiting numerous original references and we prefer to cite review articles and some few seminal papers.

In Sec. II we overview the string and the superstring theories from the perturbative point of view. T-duality and D-branes is considered in Sec. III. The Sec. IV is devoted to describe the string dualities and the web of string theories connected by duality. M and F theories are also briefly described. Sec. V is devoted to review the non-BPS branes and their description in terms of topological K-Theory. Some recent results by Witten and Moore-Witten, about the classification of Ramond-Ramond fields is also described. The relation of string theory to noncommutative Yang-Mills theory and deformation quantization theory is the theme of the Sec. VI.

II. PERTURBATIVE STRING AND SUPERSTRING THEORIES

In this section we overview some basic aspects of bosonics and fermionic strings. We focus mainly in the description of the spectrum of the theory in the light-cone gauge and the brief description of spectrum of the five consistent superstring theories (for details and further developments see for instance [1–6]).

First of all consider, as usual, the action of a relativistic point particle. It is given by $S = -m \int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu}$, where X^μ are D functions representing the coordinates of the $(D-1, 1)$-dimensional Minkowski spacetime (the target space), $\dot{X}^\mu \equiv \frac{dX^\mu}{d\tau}$ and m can be identified with the mass of the point particle. This action is proportional to the length of the world-line of the relativistic particle. In analogy with the relativistic point particle, the action describing the dynamics of a string (one-dimensional object) moving in a $(D-1, 1)$-dimensional Minkowski spacetime (the target space) is proportional to the area $\mathbf{A}$ of the worldsheet. We know from the theory of surfaces that such an area is given by $\mathbf{A} = \int \sqrt{det(-g)}$, where g is the induced metric (with signature $(-, +)$) on the worldsheet. The background metric will be denoted by $\eta_{\mu\nu}$ and $\sigma^a = (\tau, \sigma)$ with $a = 0, 1$ are the local coordinates on the worldsheet. $\eta_{\mu\nu}$ and g_{ab} are related by $g_{ab} = \eta_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$ with $\mu = 0, 1, \ldots, D-1$. Thus the classical action of a relativistic string is given by the Nambu-Goto

action

$$S_{NG}[X^\mu] = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-det(\partial_a X^\mu \partial_b X^\nu)}, \tag{1}$$

where X^μ are D embedding functions of the worldsheet into the target space X. Now introduce a metric h describing the worldsheet geometry, we get a classically equivalent action to the Nambu-Goto action. This is the Polyakov action

$$S_P[X^\mu, h_{ab}] = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}, \tag{2}$$

where the X^μ's are D scalar fields on the worldsheet. Such a fields can be interpreted as the coordinates of spacetime X (target space), $h = \det(h^{ab})$ and $h_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$.

Polyakov action has the following symmetries: (i) Poincaré invariance, (ii) Worldsheet diffeomorphism invariance, and (iii) Weyl invariance (rescaling invariance). The energy-momentum tensor of the two-dimensional theory is given by

$$T^{ab} := \frac{1}{\sqrt{-h}} \frac{\delta S_P}{\delta h_{ab}} = \frac{1}{4\pi\alpha'} \left(\partial^a X^\mu \partial^b X_\mu - \frac{1}{2} h^{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu \right) = 0. \tag{3}$$

Invariance under worldsheet diffeomorphisms implies that it should be conserved *i.e.* $\nabla_a T^{ab} = 0$, while the Weyl invariance gives the traceless condition, $T^a_a = 0$. The equation of motion associated with Polyakov action is given by

$$\partial_a \left(\sqrt{-h} h^{ab} \partial_b X^\mu \right) = 0. \tag{4}$$

Whose solutions should satisfy the boundary conditions for the open string: $\partial_\sigma X^\mu \big|_0^{\ell=\pi} = 0$ (Neumann) and for the closed string: $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$ (Dirichlet). Here $\ell = \pi$ is the characteristic length of the open string. The variation of S_P with respect to h^{ab} leads to the constraint equations $T_{ab} = 0$. From now on we will work in the *conformal gauge*. In this gauge: $h_{ab} = \eta_{ab}$ and equations of motion (4) reduces to the Laplace equation whose solutions can be written as linear superposition of plane waves.

2.1 The Closed String

For the closed string the boundary condition $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$, leads to the general solution of Eq. (4)

$$X^\mu = X_0^\mu + \frac{1}{\pi T} P^\mu \tau + \frac{i}{2\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \left\{ \alpha_n^\mu exp\left(- i2n(\tau - \sigma) \right) + \tilde{\alpha}_n^\mu exp\left(- i2n(\tau + \sigma) \right) \right\} \quad (5)$$

where X_0^μ and P^μ are the position and momentum of the center-of-mass of the string and α_n^μ and $\tilde{\alpha}_n^\mu$ satisfy the conditions $\alpha_n^{\mu*} = \alpha_{-n}^\mu$ (left-movers) and $\tilde{\alpha}_n^{\mu*} = \tilde{\alpha}_{-n}^\mu$ (right-movers).

2.2 The Open String

For the open string the respective boundary condition is $\partial_\sigma X^\mu \big|_0^{\ell=\pi} = 0$ (this is the only boundary condition which is Lorentz invariant) and the solution is given by

$$X^\mu(\tau, \sigma) = X_0^\mu + \frac{1}{\pi T} P^\mu \tau + \frac{i}{\sqrt{\pi T}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu exp(- in\tau) \cos(n\sigma) \quad (6)$$

with the condition $\alpha_n^\mu = \tilde{\alpha}_{-n}^\mu$.

2.3 Quantization

The quantization of the closed bosonic string can be carried over, as usual, by using the Dirac prescription to the center-of-mass and oscillator variables in the form

$$[X_0^\mu, P^\nu] = i\eta^{\mu\nu},$$

$$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu},$$

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0. \quad (7)$$

One can identify $(\alpha_n^\mu, \tilde{\alpha}_n^\mu)$ with the creation operators and the corresponding operators $(\alpha_{-n}^\mu, \tilde{\alpha}_{-n}^\mu)$ with the annihilation ones. In order to specify the physical states we first denote the center of mass state given by $|0; P^\mu\rangle$. The vacuum state is defined by $\alpha_m^\mu |0, P^\mu\rangle = 0$ and $P_m^\mu |0, P^\mu\rangle = p^\mu | 0, P^\mu\rangle$ and similar for the right movings. For the zero modes these states have negative norm (ghosts). However one can choice a suitable gauge where ghosts decouple from the Hilbert space when $D = 26$. This is the subject of the following subsection.

2.4 Light-cone Quantization

Now we turn out to work in the so called *light-cone gauge.* In this gauge it is possible to solve explicitly the Virasoro constraints (3). This is done by removing the light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^D)$ leaving only the transverse coordinates X^i representing the physical degrees of freedom (with $i = 1, 2, \ldots, D-2$). In this gauge the Virasoro constraints (3) are explicitly solved. Thus the independent variables are $(X_0^-, P^+, X_0^j, P^j, \alpha_n^j, \tilde{\alpha}_n^j)$. Operators α_n^- and $\tilde{\alpha}_n^-$ can be written in terms of α_n^j and $\tilde{\alpha}_n^j$ respectively as follows: $\alpha_n^- = \frac{1}{\sqrt{2\alpha'}P^+}(\sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : -2A\delta_n)$ and $\tilde{\alpha}_n^- = \frac{1}{\sqrt{2\alpha'}P^+}(\sum_{m=-\infty}^{\infty} : \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i : -2A\delta_n)$. For the open string we get $\alpha_n^- = \frac{1}{2\sqrt{2\alpha'}P^+}(\sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : -2A\delta_n)$. Here $: \cdot :$ stands for the normal ordering.

In this gauge the Hamiltonian is given by

$$H = \frac{1}{2}(P^i)^2 + N - A \quad \text{(open string)}, \qquad H = (P^i)^2 + N_L + N_R - 2A \quad \text{(closed string)} \quad (8)$$

where N is the operator number, $N_L = \sum_{m=-\infty}^{\infty} : \alpha_{-m}\alpha_m :$, and $N_R = \sum_{m=-\infty}^{\infty} : \tilde{\alpha}_{-m}\tilde{\alpha}_m :$. The *mass-shell condition* is given by $\alpha'M^2 = (N - A)$ (open string) and $\alpha'M^2 = 2(N_L + N_R - 2A)$ (closed string). For the open string, Lorentz invariance implies that the first excited state is massless and therefore $A = 1$. In the light-cone gauge A takes the form $A = -\frac{D-2}{2}\sum_{n=1}^{\infty} n$. From the fact $\sum_{n=1}^{\infty} n^{-s} = \zeta(s)$, where ζ is the Riemann's zeta function (which converges for $s > 1$ and has a unique analytic continuation at $s = -1$, where it takes the value $-\frac{1}{12}$) then $A = -\frac{D-2}{24}$ and therefore $D = 26$.

2.5 Spectrum of the Bosonic String

Closed Strings

The spectrum of the closed string can be obtained from the combination of the left-moving states and the the right-moving ones. The ground state ($N_L = N_R = 0$) is given by $\alpha'M^2 = -4$. That means that the ground state includes a tachyon. The first excited state ($N_L = 1 = N_R$) is massless and it is given by $\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, P\rangle$. This state can be naturally decomposed into irreducible representations of the little group $SO(24)$ as follows

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}\mid 0, P\rangle = \alpha^{[i}_{-1}\tilde{\alpha}^{j]}_{-1}\mid 0, P\rangle + \left(\alpha^{(i}_{-1}\tilde{\alpha}^{j)}_{-1} - \frac{1}{D-2}\delta^{ij}\alpha^k_{-1}\tilde{\alpha}^k_{-1}\right)\mid 0, P\rangle$$

$$+ \ \frac{1}{D-2}\delta^{ij}\alpha^k_{-1}\tilde{\alpha}^k_{-1}\mid 0, P\rangle. \tag{9}$$

The first term of the rhs is interpreted as a spin 2 massless particle g_{ij} (*graviton*). The second term is a range 2 anti-symmetric tensor B_{ij}. While the last term is an scalar field ϕ (*dilaton*). Higher excited massive states are combinations of representations of the little group SO(25).

Open Strings

For the open string, the ground state includes once again a tachyon since $\alpha'M^2 = -1$. The first exited state $N = 1$ is given by a massless vector field in 26 dimensions. The second excitation level is given by the massive states $\alpha^i_{-2}\mid 0, P\rangle$ and $\alpha^i_{-1}\alpha^j_{-1}\mid 0, P\rangle$ which are in irreducible representations of the little group SO(25).

2.6 Superstrings

In bosonic string theory there are two big problems. The first one is the presence of tachyons in the spectrum. The second one is that there are no spacetime fermions. Here is where superstrings come to the rescue. A superstring is described, despite of the usual bosonic fields X^μ, by fermionic fields $\psi^\mu_{L,R}$ on the worldsheet. Which satisfy anticommutation rules and where the L and R denote the left and right worldsheet chirality respectively. The action for the superstring is given by

$$L_{SS} = -\frac{1}{8\pi}\int d^2\sigma\sqrt{-h}\left(h^{ab}\partial_a X^\mu\partial_b X_\mu + 2i\bar{\psi}^\mu\gamma^a\partial_a\psi_\mu - i\bar{\chi}_a\gamma^b\gamma^a\psi^\mu(\partial_b X_\mu - \frac{i}{4}\bar{\chi}_b\psi_\mu)\right), \tag{10}$$

where ψ^μ and χ_a are the superpartners of X^μ and the tetrad field e^a, respectively. In the superconformal gauge and in light-cone coordinates it can be reduced to

$$L_{SS} = \frac{1}{2\pi}\int\left(\partial_L X^\mu\partial_R X_\mu + i\psi^\mu_R\partial_L\psi_{\mu R} + i\psi^\mu_L\partial_R\psi_{\mu R}\right). \tag{11}$$

In analogy to the bosonic case, the local dynamics of the worldsheet metric is manifestly independent of quantum corrections if the critical spacetime dimension D is 10. Thus the string oscillates in the 8 transverse dimensions. The action (10) is invariant under: (i) worldsheet supersymmetry, (ii) Weyl transformations, (iii) super-Weyl transformations, (iv) Poincaré transformations and (v) Worldsheet reparametrizations. The equation of motion for the $X's$ fields is the same that in the bosonic case (Laplace equation) and whose general solution is given by Eqs. (5) or (6). Equation of motion for the fermionic field is the Dirac equation in two dimensions. Constraints here are more involved and they are called the *super-Virasoro constraints*. However in the light-cone gauge, everything simplifies and the transverse coordinates (eight coordinates) become the bosonic physical degrees of freedom together with their corresponding supersymmetric partners. Analogously to the bosonic case, massless states of the spectrum come into representations of the little group SO(8) of SO(9, 1), while that the massive states lie into representations of the little group SO(9).

For the closed string there are two possibilities for the boundary conditions of fermions: (i) periodic boundary conditions (Ramond (**R**) sector) $\psi_{L,R}^{\mu}(\sigma) = +\psi_{L,R}^{\mu}(\sigma + 2\pi)$ and (ii) anti-periodic boundary conditions (Neveu-Schwarz (**NS**) sector) $\psi_{L,R}^{\mu}(\sigma) = -\psi_{L,R}^{\mu}(\sigma + 2\pi)$. Solutions of Dirac equation satisfying these boundary conditions are

$$\psi_L^{\mu}(\sigma, \tau) = \sum_n \bar{\psi}_{-n}^{\mu} exp\Big(-in(\tau + \sigma)\Big), \qquad \psi_R^{\mu}(\sigma, \tau) = \sum_n \psi_n^{\mu} exp\Big(-in(\tau - \sigma)\Big). \qquad (12)$$

In the case of the fermions in the **R** sector n is integer and it is semi-integer in the **NS** sector.

The quantization of the superstring come from the promotion of the fields X^{μ} and ψ^{μ} to operators whose oscillator variables are operators satisfying the relations $[\alpha_n^{\mu}, \alpha_m^{\nu}]_- = n\delta_{m+n,0}\eta^{\mu\nu}$ and $[\psi_n^{\mu}, \psi_m^{\nu}]_+ = \eta^{\mu\nu}\delta_{m+n,0}$, where $[,]_-$ and $[,]_+$ stand for commutator and anti-commutator respectively.

The zero modes of α are diagonal in the Fock space and its eigenvalue can be identified with its momentum. For the **NS** sector there is no fermionic zero modes but they can exist for the **R** sector and they satisfy a Clifford algebra $[\psi_0^{\mu}, \psi_0^{\nu}]_+ = \eta^{\mu\nu}$. The Hamiltonian for the

closed superstring is given by $H_{L,R} = N_{L,R} + \frac{1}{2}P_{L,R}^2 - A_{L,R}$. For the **NS** sector $A = \frac{1}{2}$, while for the **R** sector $A = 0$. The mass is given by $M^2 = M_L^2 + M_R^2$ with $\frac{1}{2}M_{L,R}^2 = N_{L,R} - A_{L,R}$.

There are five consistent superstring theories: Type IIA, IIB, Type I, SO(32) and $E_8 \times E_8$ heterotic strings. In what follows of this section we briefly describe the spectrum in each one of them.

2.7 Type II Superstring Theories

In this case the theory consist of closed strings only. They are theories with $\mathcal{N} = 2$ spacetime supersymmetry. For this reason, there are 8 scalar fields (representing the 8 transverse coordinates to the string) and one Weyl-Majorana spinor. There are 8 left-moving and 8 right-moving fermions.

In the **NS** sector there is still a tachyon in the ground state. But in the supersymmetric case this problem can be solved through the introduction of the called **GSO** projection. This projection eliminates the tachyon in the **NS** sector and it acts in the **R** sector as a ten-dimensional spacetime chirality operator. That means that the application of the **GSO** projection operator defines the chirality of a massless spinor in the **R** sector. Thus from the left and right moving sectors, one can construct states in four different sectors: (i) **NS-NS**, (ii) **NS-R**, (iii) **R-NS** and (iv) **R-R**. Taking account the two types of chirality L and R one has two possibilities:

$a)-$ The **GSO** projections on the left and right fermions produce different chirality in the ground state of the **R** sector (*Type IIA*).

$b)-$**GSO** projection are equal in left and right sectors and the ground states in the **R** sector, have the same chirality (*Type IIB*). Thus the spectrum for the Type IIA and IIB superstring theories is:

Type IIA

The **NS-NS** sector has a symmetric tensor field $g_{\mu\nu}$ (spacetime metric), an antisymmetric

tensor field $B_{\mu\nu}$ and a scalar field ϕ (dilaton). In the **R-R** sector there is a vector field A_μ associated with a 1-form C_1 ($A_\mu \Leftrightarrow C_1$) and a rank 3 totally antisymmetric tensor $C_{\mu\nu\rho} \Leftrightarrow C_3$. In general the **R-R** sector consist of p-forms $G_p = dC_{p-1}$ (where C_p are called RR fields) on the ten-dimensional spacetime X with p even $i.e.$ G_0, G_2, G_4, $\ldots$. In the **NS-R** and **R-NS** sectors we have two gravitinos with opposite chirality and the supersymmetric partners of the mentioned bosonic fields.

Type IIB

In the **NS-NS** sector Type IIB theory has exactly the same spectrum that of Type IIA theory. On the **R-R** sector it has a scalar field $\chi \Leftrightarrow C_0$, an antisymmetric tensor field $B'_{\mu\nu} \Leftrightarrow C_2$ and a rank 4 totally antisymmetric tensor $D_{\mu\nu\rho\sigma} \Leftrightarrow C_4$ whose field strength is self-dual $i.e.$, $G_5 = dC_4$ with $*G_5 = +G_5$. Similar than for the case of Type IIA theory one has, in general, RR fields given by p-forms $G_p = dC_{p-1}$ on the spacetime X with p odd $i.e.$ G_1, G_3, G_5, $\ldots$. The **NS-R** and **R-NS** sectors do contain gravitinos with the same chirality and the corresponding fermionic matter.

2.8 Type I Superstrings

In this case the L and R degrees of freedom are the same. Type I and Type IIB theories have the same spectrum, except that in the former one the states which are not invariant under the change of orientation of the worldsheet, are projected out. This worldsheet parity Ω interchanges the left and right modes. Type I superstring theory is a theory of breakable closed strings, thus it incorporates also open strings. The Ω operation leave invariant only one half of the spacetime supersymmetry, thus the theory is $\mathcal{N} = 1$.

The spectrum of bosonic massless states in the **NS-NS** sector is: $g_{\mu\nu}$ (spacetime metric) and ϕ (dilaton) from the closed sector and $B_{\mu\nu}$ is projected out. On the **R-R** sector there is an antisymmetric field $B_{\mu\nu}$ of the closed sector. The open string sector is necessary in order to cancel tadpole diagrams. A contribution to the spectrum come from this sector.

Chan-Paton factors can be added at the boundaries of open strings. Hence the cancellation of the tadpole are needed 32 labels at each end. Therefore in the **NS-NS** sector there are 496 gauge fields in the adjoint representation of SO(32).

2.9 Heterotic Superstrings

This kind of theory involves only closed strings. Thus there are left and right sectors. The left-moving sector contains a bosonic string theory and the right-moving sector contains superstrings. This theory is supersymmetric on the right sector only, thus the theory contains $\mathcal{N} = 1$ spacetime supersymmetry. The momentum at the left sector P_L lives in 26 dimensions, while P_R lives in 10 dimensions. It is natural to identify the first ten components of P_L with P_R. Consistency of the theory tell us that the extra 16 dimensions should belong to the root lattice $E_8 \times E_8$ or a $\mathbf{Z}_2$-sublattice of the SO(32) weight lattice.

The spectrum consists of a tachyon in the ground state of the left-moving sector. In both sectors we have the spacetime metric $g_{\mu\nu}$, the antisymmetric tensor $B_{\mu\nu}$, the dilaton ϕ and finally there are 496 gauge fields A_μ in the adjoint representation of the gauge group $E_8 \times E_8$ or SO(32).

III. TOROIDAL COMPACTIFICATION, T-DUALITY AND D-BRANES

D-branes are, despite of the dual fundamental degrees of freedom in string theory, extremely interesting and useful tools to study nonperturbative properties of string and field theories (for a classic review see [7]). Non-perturbative properties of supersymmetric gauge theories can be better understanding as the world-volume effective theory of some configurations of intersecting D-branes (for a review see [8]). D-branes also are very important to connect gauge theories with gravity. This is the starting point of the AdS/CFT correspondence or Maldacena's conjecture. We don't review this interesting subject in this paper,

however the reader can consult the excellent review [9]. Roughly speaking D-branes are static solutions of string equations which satisfy Dirichlet boundary conditions. That means that open strings can end on them. To explain these objects we follow the traditional way, by using T-duality on open strings we will see that Neumann conditions are turned out into the Dirichlet ones. To motivate the subject we first consider T-duality in closed bosonic string theory.

3.1 T-duality in Closed Strings

The general solution of Eq. (4) in the conformal gauge can be written as $X^\mu(\sigma, \tau) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+)$, where $\sigma^\pm = \sigma \pm \tau$. Now, take one coordinate, say X^{25} and compactify it on a circle of radius R. Thus we have that X^{25} can be identified with $X^{25} + 2\pi R m$ where m is called the *winding number*. The general solution for X^{25} with the above compactification condition is

$$X_R^{25}(\sigma^-) = X_{0R}^{25} + \sqrt{\frac{\alpha'}{2}} P_R^{25}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} \alpha_{R,l}^{25} exp\Big(- il(\tau - \sigma)\Big)$$

$$X_L^{25}(\sigma^+) = X_{0L}^{25} + \sqrt{\frac{\alpha'}{2}} P_L^{25}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{l} \alpha_{L,l}^{25} exp\Big(- il(\tau + \sigma)\Big), \qquad (13)$$

where

$$P_{R,L}^{25} = \frac{1}{\sqrt{2}}\Big(\frac{\sqrt{\alpha'}}{R} n \mp \frac{R}{\sqrt{\alpha'}} m\Big). \qquad (14)$$

Here n and m are integers representing the discrete momentum and the winding number, respectively. The latter has not analogous in field theory. While the canonical momentum is given by $P^{25} = \frac{1}{\sqrt{2\alpha'}}(P_L^{25} + P_R^{25})$. Now, by the mass shell condition, the mass of the perturbative states is given by $M^2 = M_L^2 + M_R^2$, with

$$M_{L,R}^2 = -\frac{1}{2} P^\mu P_\mu = \frac{1}{2}(P_{L,R}^{25})^2 + \frac{2}{\alpha'}(N_{L,R} - 1). \qquad (15)$$

We can see that for all states with $m \neq 0$, as $R \to \infty$ the mass become infinity, while $m = 0$ implies that the states take all values for n and form a continuum. At the case

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when $R \to 0$, for states with $n \neq 0$, mass become infinity. However in the limit $R \to 0$ for $n = 0$ states with all m values produce a continuum in the spectrum. So, in this limit the compactified dimension disappears. For this reason, we can say that the mass spectrum of the theories at radius R and $\frac{\alpha'}{R}$ are identical when we interchange $n \Leftrightarrow m$. This symmetry is known as *T-duality*.

The importance of T-duality lies in the fact that the T-duality transformation is a parity transformation acting on the left and right moving degrees of freedom. It leaves invariant the left movers and changes the sign of the right movers (see Eq. (14))

$$P_L^{25} \to P_L^{25}, \qquad P_R^{25} \to -P_R^{25}. \tag{16}$$

The action of T-duality transformation must leave invariant the whole theory (at all order in perturbation theory). Thus, all kind of interacting states in certain theory should correspond to those states belonging to the dual theory. In this context, also the vertex operators are invariant. For instance the tachyonic vertex operators are

$$V_L = exp(iP_L^{25}X_L^{25}), \qquad V_R = exp(iP_R^{25}X_R^{25}). \tag{17}$$

Under T-duality, $X_L^{25} \to X_L^{25}$ and $X_R^{25} \to -X_R^{25}$; and from the general solution Eq. (13), $\alpha_{R,i}^{25} \to -\alpha_{R,i}^{25}$, $X_{0R}^{25} \to -X_{0R}^{25}$. Thus, T-duality interchanges $n \Leftrightarrow m$ (Kaluza-Klein modes $\Leftrightarrow$ winding number) and $R \Leftrightarrow \frac{\alpha'}{R}$ in closed string theory.

3.2 T-duality in Open Strings

Now, consider *open strings* with Neumann boundary conditions. Take again the 25th coordinate and compactify it on a circle of radius R, but keeping Neumann conditions. As in the case of closed string, center of mass momentum takes only discrete values $P^{25} = \frac{n}{R}$. While there is not analogous for the winding number. So, when $R \to 0$ all states with nonzero momentum go to infinity mass, and do not form a continuum. This behavior is similar as in field theory, but now there is something new. The general solutions are

$$X_R^{25} = \frac{X_0^{25}}{2} - \frac{a}{2} + \alpha' P^{25}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} \alpha_l^{25} exp\Big(-i2l(\tau - \sigma)\Big),$$

$$X_L^{25} = \frac{X_0^{25}}{2} + \frac{a}{2} + \alpha' P^{25}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{l \neq 0} \frac{1}{l} \alpha_l^{25} exp\Big(-i2l(\tau + \sigma)\Big) \qquad (18)$$

where a is a constant. Thus, $X^{25}(\sigma, \tau) = X_R^{25}(\sigma^-) + X_L^{25}(\sigma^+) = X_0^{25} + \frac{2\alpha'n}{R}\tau +$ oscillator terms. Taking the limit $R \to 0$, only the $n = 0$ mode survives. Because of this, the string seems to move in 25 spacetime dimensions. In other words, the strings vibrate in 24 transversal directions. T-duality provides a new T-dual coordinate defined by $\tilde{X}^{25}(\sigma, \tau) = X_L^{25}(\sigma, \tau) - X_R^{25}(\sigma, \tau)$. Now, taking $\tilde{R} = \frac{\alpha'}{R}$ we have $\tilde{X}^{25}(\sigma, \tau) = a + 2\tilde{R}\sigma n + $ oscillator terms. Using the boundary conditions at $\sigma = 0, \pi$ one has $\tilde{X}^{25}(\sigma, \tau)\mid_{\sigma=0} = a$ and $\tilde{X}^{25}(\sigma, \tau)\mid_{\sigma=\pi} = a + 2\pi\tilde{R}n$. Thus, we started with an open bosonic string theory with Neumann boundary conditions, and T-duality and a compactification on a circle in the 25th dimension, give us Dirichlet boundary conditions in such a coordinate. We can visualize this saying that an open string has its endpoints fixed at a hyperplane with 24 dimensions.

Strings with $n = 0$ lie on a 24 dimensional plane space (D24-brane). Strings with $n = 1$ has one endpoint at a hyperplane and the other at a different hyperplane which is separated from the first one by a factor equal to $2\pi\tilde{R}$, and so on. But if we compactify p of the X^i directions over a T^p torus $(i = 1, ..., p)$. Thus, after T-dualizing them we have strings with endpoints fixed at hyperplane with $25 - p$ dimensions, the D$(25 - p)$-brane.

Summarizing: the system of open strings moving freely in spacetime with p compactified dimensions on T^p is equivalent, under T-duality, to strings whose enpoints are fixed at a D(25-p)-brane *i.e.* obeying Neumann boundary conditions in the X^i longitudinal directions $(i = 1, \ldots, p)$ and Dirichlet ones in the transverse coordinates X^m $(m = p + 1, ..., 25)$.

The effect of T-dualizing a coordinate is to change the nature of the boundary conditions, from Neumann to Dirichlet and viceverse. If one dualize a longitudinal coordinate this coordinate will satisfies the Dirichlet condition and the Dp-brane becomes a D$(p+1)$-brane.

But if the dualized coordinate is one of the transverse coordinates the Dp-brane becomes a D$(p-1)$-brane.

T-duality also acts conversely. We can think to begin with a closed string theory, and compactify it on to a circle in the 25th coordinate, and then by imposing Dirichlet conditions, obtain a D-brane. This is precisely what occurs in Type II theory, a theory of closed strings.

Spectrum and Wilson Lines

Now, we will see how emerges a gauge field on the Dp-brane world-volume. Again, for the mass shell condition for open bosonic strings and because T-duality $M^2 = (\frac{n}{\alpha'}\tilde{R})^2 + \frac{1}{\alpha'}(N-1)$. The massless state ($N = 1$, $n = 0$) implies that the gauge boson $\alpha^{\mu}_{-1} \mid 0\rangle$ ($U(1)$ gauge boson) lies on to the D24-brane world-volume. On the other hand, $\alpha^{25}_{-1} \mid 0\rangle$ has a *vev* (vacuum expectation value) which describes the position $\tilde{X}^{25}$ of the D-brane after T-dualizing. Thus, we can say in general, there is a gauge theory $U(1)$ over the world volume of the Dp-brane.

Consider now an *orientable open string*. The endpoints of the string carry charge under a non-Abelian gauge group. For Type II theories the gauge group is $U(N)$. One endpoint transforms under the fundamental representation **N** of $U(N)$ and the other one, under its complex conjugate representation (the anti-fundamental one) $\bar{\mathbf{N}}$.

The ground state wave function is specified by the center of mass momentum and by the charges of the endpoints. Thus implies the existence of a basis $\mid k; ij\rangle$ called *Chan-Paton basis*. States $\mid k; ij\rangle$ of the Chan-Paton basis are those states which carry charge 1 under the ith $U(1)$ generator and -1 under the jth $U(1)$ generator. So, we can decompose the wave function for ground state as $\mid k; a\rangle = \sum_{i,j=1}^{N} \mid k; ij\rangle\lambda^a_{ij}$ where λ^a_{ij} are called *Chan-Paton factors*. From this, we see that it is possible to add degrees of freedom to endpoints of the string, that are precisely the Chan-Paton factors.

This is consistent with the theory, because the Chan-Paton factors have a Hamiltonian which do not posses dynamical structure. So, if one endpoint to the string is prepared in a certain state, it always will remains the same. It can be deduced from this, that

$\lambda^a \longrightarrow U\lambda^a U^{-1}$ with $U \in U(N)$. Thus, the worldsheet theory is symmetric under $U(N)$, and this global symmetry is a gauge symmetry in spacetime. So the vector state at massless level $\alpha^\mu_{-1} \mid k, a\rangle$ is a $U(N)$ gauge boson.

When we have a gauge configuration with non trivial line integral around a compactified dimension (i.e a circle), we said there is a Wilson line. In case of open strings with gauge group $U(N)$, a toroidal compactification of the 25th dimension on a circle of radius R. If we choice a background field A^{25} given by $A^{25} = \frac{1}{2\pi R} diag(\theta_1, ..., \theta_N)$ a Wilson line appears. Moreover, if $\theta_i = 0$, $i = 1, ..., l$ and $\theta_j \neq 0$, $j = l + 1, ..., N$ then gauge group is broken: $U(N) \longrightarrow U(l) \times U(1)^{N-l}$. It is possible to deduce that θ_i plays the role of a Higgs field. Because string states with Chan-Paton quantum numbers $\mid ij\rangle$ have charges 1 under i^{th} $U(1)$ factor (and -1 under j^{th} $U(1)$ factor) and neutral with all others; canonical momentum is given now by $P^{25}_{(ij)} \Longrightarrow \frac{n}{R} + \frac{(\theta_j - \theta_i)}{2\pi R}$. Returning to the mass shell condition it results,

$$M^2_{ij} = \left(\frac{n}{R} + \frac{\theta_j - \theta_i}{2\pi R}\right)^2 + \frac{1}{\alpha'}(N - 1). \tag{19}$$

Massless states ($N = 1, n = 0$) are those in where $i = j$ (diagonal terms) or for which $\theta_j = \theta_i$ ($i \neq j$). Now, T-dualizing we have $\tilde{X}^{25}_{ij}(\sigma, \tau) = a + (2n + \frac{\theta_j - \theta_i}{\pi})\tilde{R}\sigma + oscillator\ terms$. Taking $a = \theta_i\tilde{R}$, $\tilde{X}^{25}_{ij}(0, \tau) = \theta_i\tilde{R}$ and $\tilde{X}^{25}_{ij}(\pi, \tau) = 2\pi n\tilde{R} + \theta_j\tilde{R}$. This give us a set of N D-branes whose positions are given by $\theta_j\tilde{R}$, and each set is separated from its initial positions ($\theta_j = 0$) by a factor equal to $2\pi\tilde{R}$. Open strings with both endpoints on the same D-brane gives massless gauge bosons. The set of N D-branes give us $U(1)^N$ gauge group. An open string with one endpoint in one D-brane, and the other endpoint in a different D-brane, yields a massive state with $M \sim (\theta_j - \theta_i)\tilde{R}$. Mass decreases when two different D-branes approximate to each other, and are null when become the same. When all D-branes take up the same position, the gauge group is enhanced from $U(1)^N$ to $U(N)$. On the D-brane world-volume there are also scalar fields in the adjoint representation of the gauge group $U(N)$. The scalars parametrize the transverse positions of the D-brane in the target space X.

3.3 D-brane Actions and Ramond-Ramond Charges

D-Brane Action

With the massless spectrum on the D-brane world-volume it is possible to construct a low energy effective action. For open strings massless fields are interacting with the closed strings massless spectrum from the **NS-NS** sector. Let ξ^a with $a = 0, \ldots, p$ the wold-volume coordinates. The effective action is the gauge invariant action well known as the Born-Infeld action

$$S_D = -T_p \int_W d^{p+1}\xi e^{-\Phi}\sqrt{det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})}, \tag{20}$$

where T_p is the tension of the D-brane, G_{ab} is the world-volume induced metric, B_{ab} is the induced antisymmetric field, F_{ab} is the Abelian field strength on W and Φ is the dilaton field.

For N D-branes the massless fields turns out to be $N \times N$ matrices and the action turns out to be non-Abelian Bon-Infeld action (for a nice review about the Born-Infeld action in string theory see [10])

$$S_D = -T_p \int_W d^{p+1}\xi e^{-\Phi} Tr\left(\sqrt{det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + O([X^m, X^n]^2)\right) \tag{21}$$

where $m, n = p + 1, \ldots, 10$. The scalar fields X^m representing the transverse positions become $N \times N$ matrices and so, the spacetime become a noncommutative spacetime. We will come back later to this interesting point.

Ramond-Ramond Charges

D-branes are coupled to Ramond-Ramond (RR) fields G_p. The complete effective action on the D-brane world-volume W which take into account this coupling is

$$S_D = -T_p \int_W d^{p+1}\xi\left\{e^{-\Phi}\sqrt{det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} + i\mu_p \int_W \sum_p C_{(p+1)} Tr\left(e^{2\pi\alpha'(F+B)}\right)\right\} \tag{22}$$

where μ_p us the RR charge. RR charges can be computed by considering the anomalous behavior of the action at intersections of D-branes [11]. Thus RR charge is given by

$$Q_{RR} = ch(j!E)\sqrt{\hat{A}(TX)} \tag{23}$$

where $j : W \hookrightarrow X$. Here E is the Chan-Paton bundle over X, $\hat{A}(TX)$ is the genus of the spacetime manifold X. This gives an ample evidence that the RR charges take values not in a cohomology theory, but in fact, in a K-Theory [11]. This result was further developed by Witten in the context of non-BPS brane configurations worked out by A. Sen. This subject will be reviewed below in Sec. V.

IV. STRING DUALITY

4.1 Duality in Field Theory

Duality is a notion which in the last years has led to remarkable advances in nonperturbative quantum field theory. It is an old known type of symmetry which by interchanging the electric and magnetic fields leaves invariant the vacuum Maxwell equations. It was extended by Dirac to include sources, with the well known price of the prediction of monopoles, which appear as the dual particles to the electrically charged ones and whose existence could not be confirmed up to now. Dirac obtained that the couplings (charges) of the electrical and magnetical charged particles are the inverse of each other, *i.e.* as the electrical force is 'weak', and it can be treated perturbatively, then the magnetic force among monopoles will be 'strong' (for some reviews see [12–14]).

This duality, called 'S-duality', has inspired a great deal of research in the last years. By this means, many non-perturbative exact results have been established. In particular, the exact Wilson effective action of $\mathcal{N} = 2$ supersymmetric gauge theories has been computed by Seiberg and Witten, showing the duality symmetries of these effective theories. It turns out that the dual description is quite adequate to address standard non-perturbative problems of Yang-Mills theory, such as confinement, chiral symmetry breaking, etc.

$\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions have vanishing renormalization

group β-function. Montonen and Olive conjectured that (at the quantum level) these theories would possess an $\mathrm{SL}(2, \mathbf{Z})$ exact dual symmetry. Many evidences of this fact have been found, although a rigorous proof does not exist at present. For $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions, the β-function in general does not vanish. So, Montonen-Olive conjecture cannot be longer valid in the same sense as for $\mathcal{N} = 4$ theories. However, Seiberg and Witten found that a strong-weak coupling 'effective duality' can be defined on its low energy effective theory for the cases pure and with matter. The quantum moduli space of the pure theory is identified with a complex plane, the u-plane, with singularities located at the points $u = \pm 1, \infty$. It turns out that at $u = \pm 1$ the original Yang-Mills theory is strongly coupled, but effective duality permits the weak coupling description at these points in terms of monopoles or dyons (dual variables). $\mathcal{N} = 1$ gauge theories are also in the class of theories with non-vanishing β-function. More general, for a gauge group $\mathrm{SU}(N_c)$, an effective non-Abelian duality is implemented even when the gauge symmetry is unbroken. It has a non-Abelian Coulomb phase. Seiberg has shown that this non-Abelian Coulomb branch is dual to another non-Abelian Coulomb branch of a theory with gauge group $\mathrm{SU}(N_f - N_c)$, where N_f is the number of flavors. $\mathcal{N} = 1$ theories have a rich phase structure. Thus, it seems that in supersymmetric gauge theories strong-weak coupling duality can only be defined for some particular phases.

For non-supersymmetric gauge theories in four dimensions, the subject of duality has been explored recently in the Abelian as well as in the non-Abelian cases. In the Abelian case (on a curved compact four-manifold X) the CP violating Maxwell theory partition function $Z(\tau)$, transforms as a *modular form* under a finite index subgroup $\Gamma_0(2)$ of $\mathrm{SL}(2, \mathbf{Z})$. The dependence parameter of the partition function is given by $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$, where e is the Abelian coupling constant and θ is the usual theta angle. In the case of non-Abelian non-supersymmetric gauge theories, strong-coupling dual theories can be constructed which results in a kind of dual "massive" non-linear sigma models. The starting Yang-Mills theory contains a CP-violating θ-term and it turns out to be equivalent to the linear combination of the actions corresponding to the self-dual and anti-self-dual field strengths.

4.2 String Duality

In Sec. I we have described the massless spectrum of the five consistent superstring theories in ten dimensions. Additional theories can be constructed in lower dimensions by compactification of some of the ten dimensions. Thus the ten-dimensional spacetime X looks like the product $X = K^d \times \mathbf{R}^{1,9-d}$, with K a suitable compact manifold or orbifold. Depending on which compact space is taken, it will be the quantity of preserved supersymmetry.

All five theories and their compactifications are parametrized by: the string coupling constant g_S, the geometry of the compact manifold K, the topology of K and the spectrum of bosonic fields in the **NS-NS** and the **R-R** sectors. Thus one can define the *string moduli space* of each one of the theories as the space of all associated parameters. Moreover, it can be defined a map between two of these moduli spaces. The dual map is defined as the map $\mathcal{S} : \mathcal{M} \to \mathcal{M}'$ between the moduli spaces $\mathcal{M}$ and $\mathcal{M}'$ such that the strong coupling region of $\mathcal{M}$ is interchanged with the weak-coupling region of $\mathcal{M}'$ and viceverse. One can define another map $\mathcal{T} : \mathcal{M} \to \mathcal{M}'$ which interchanges the volume V of K for $\frac{1}{V}$. One example of the map $\mathcal{T}$ is the equivalence, by T-duality, between the theories Type IIA compactified on $\mathbf{S}^1$ at radius R and the Type IIB theory on $\mathbf{S}^1$ at raduis $\frac{1}{R}$. The theories $\mathrm{Het}(E_8 \times E_8)$ and $\mathrm{Het}(\mathrm{SO}(32))$ is another example. In this section we will follows the Sen's review [15]. Another useful reviews are [16–19].

4.3 Type I-SO(32)-Heterotic Duality

In order to analyze the duality between Type I and SO(32) heterotic string theories we recall from Sec. II the spectrum of both theories. These fields are the dynamical fields of a supergravity Lagrangian in ten dimensions. Type I string theory has in the **NS-NS** sector the fields: the metric $g^I_{\mu\nu}$, the dilaton Φ^I and in the **R-R** sector: the antisymmetric tensor $B^I_{\mu\nu}$. Also there are 496 gauge bosons A^{aI}_μ in the adjoint representation of the gauge group SO(32). For the SO(32) heterotic string theory the spectrum consist of: the spacetime

metric $g_{\mu\nu}^H$, the dilaton field Φ^H, the antisymmetric tensor $B_{\mu\nu}^H$ and 496 gauge fields A_μ^{aH} in the adjoint representation of SO(32). Both theories have spacetime supersymmetry $\mathcal{N} = 1$. The effective action for the massless fields of the Type I supergravity effective action S_I is defined at tree-level on the disk. Thus the string coupling constant g_s^I arises in the Einstein frame as $exp(-\Phi^I/4)$. While the heterotic action S_H is defined on the sphere and g_S^H is given by $exp(\Phi^H/4)$. The comparison of these two actions in the Einstein frame leads to the following identification of the fields

$$g_{\mu\nu}^I = g_{\mu\nu}^H, \qquad B_{\mu\nu}^I = B_{\mu\nu}^H$$

$$A_\mu^{aI} = A_\mu^{aH}, \qquad \Phi^I = -\Phi^H. \tag{24}$$

This give us many information, the first relation tell us that the metrics of both theories are the same. The second relation interchanges the B fields in the **NS-NS** and the **R-R** sectors. That interchanges heterotic strings by Type I D1-branes. The third relation identifies the gauge fields coming from the Chan-Paton factors from the Type I side with the gauge fields coming from the 16 compactified internal dimensions of the heterotic string. Finally, the opposite sign for the dilaton relation means that the string coupling constant g_S^I is inverted $g_S^H = 1/g_S^I$ within this identification, and interchanges the strong and weak couplings of both theories leading to the explicit realization of the $\mathcal{S}$ map.

4.4 Type II-Heterotic Duality

Lower dimensional theories constructed up on compactification can have different spacetime supersymmetry. Thus it can be very useful to find dual pairs by compactifying two string theories with different spacetime supersymmetry on different spaces K in such a way that they become to have the same spacetime supersymmetry.

Perhaps the most famous example is the S-dual pair between the Type II theory on $K3$ and the heterotic theory on T^4. To describe more generally these kind of dualities we first give some preliminaries. Let $\mathcal{A}$ and $\mathcal{B}$ two different theories of the family of string theories.

$\mathcal{A}$ and $\mathcal{B}$ are compactified on $K_{\mathcal{A}}$ and $K_{\mathcal{B}}$ respectively. Consider the dual pair

$$\mathcal{A}/K_{\mathcal{A}} \Longleftrightarrow \mathcal{B}/K_{\mathcal{B}} \tag{25}$$

then we can construct the more general dual pair

$$\mathcal{A}/Q_{\mathcal{A}} \Longleftrightarrow \mathcal{B}/Q_{\mathcal{B}}, \tag{26}$$

where $K_{\mathcal{A}} - Q_{\mathcal{A}} \to D$ and $K_{\mathcal{B}} - Q_{\mathcal{B}} \to D$ are fibrations and D is an auxiliary finite dimensional manifold.

These insights are very useful to construct dual pairs for theories with eight supercharges. An example of this is the pair in six dimensions with $\mathcal{A} = IIA$, $K_{\mathcal{A}} = K3$ and $\mathcal{B} = Het$, $K_{\mathcal{B}} = T^4$ *i.e.*

$$IIA/K3 \Longleftrightarrow Het/T^4. \tag{27}$$

From this a dual pair can be constructed in four dimensions with the auxiliary space $D = \mathbf{CP}^1$ being the complex projective space, thus we have

$$IIA/CY \Longleftrightarrow Het/K3 \times T^2, \tag{28}$$

where $T^4 - Q_{IIA} \to \mathbf{CP}^1$ and $K3 - Q_{Het} \to \mathbf{CP}^1$ are fibrations. As can be observed the four-dimensional theories have $\mathcal{N} = 2$ supersymmetry and the duality uses K3-fibrations.

4.5 M-Theory

We have described how to construct dual pairs of string theories. By the uses of the $\mathcal{S}$ and the $\mathcal{T}$ maps a network of theories can be constructed in various dimensions all of them related by dualities. However new theories can emerge from this picture, this is the case of M-theory. M-theory (the name come from 'mystery', 'magic', 'matrix', 'membrane', etc.) was originally defined as the strong coupling limit for Type IIA string theory [20]. At the effective low energy action level, Type IIA theory is described by the Type IIA supergravity theory and it is known that this theory can be obtained from the dimensional reduction of

the eleven dimensional supergravity theory (a theory known from the 70's years). Let Y be the eleven dimensional manifold, taking $Y = X^{10} \times \mathbf{S}^1_R$ the compactification radius R is proportional to $g_{10,10} \equiv \Phi$. Thus the limit $\Phi \to \infty$ corresponds to the limit $R \to \infty$ and thus the strong coupling limit of the Type IIA theory corresponds to the 11 dimensional supergravity. It is conjectured that there exist an eleven dimensional fundamental theory whose low energy limit is the 11 dimensional supergravity theory. At the present time the degrees of freedom are still unknown, through at the macroscopic level they should be membranes and fivebranes (also called M-two-branes and M-fivebranes). There is a proposal to describe dof of M-theory in terms of a gas of D0-branes. This is the called 'Matrix Theory'. This proposal as been quite successful (for some reviews see [21,22] and references therein).

4.6 Horava-Witten Theory

Just as the M-theory compactification on $\mathbf{S}^1_R$ leads to the Type IIA theory, Horava and Witten realized that orbifold compactifications leads to the $E_8 \times E_8$ heterotic theory in ten dimensions (see for instance [17]). More precisely

$$\mathrm{M}/\mathbf{S}^1/\mathbf{Z}_2 \Longleftrightarrow E_8 \times E_8 \; Het \tag{29}$$

where $\mathbf{S}^1/\mathbf{Z}_2$ is homeomorphic to the finite interval I and the M-theory is thus defined on $Y = X^{10} \times I$. From the ten-dimensional point of view, this configuration is seen as two parallel planes placed at the two boundaries ∂I of I. Dimensional reduction and anomalies cancellation conditions imply that the gauge degrees of freedom should be trapped on the ten-dimensional planes X with the gauge group being E_8 in each plane. While that the gravity is propagating in the bulk and thus both copies of X's are only connected gravitationally.

4.7 F-Theory

F-Theory was formulated by C. Vafa, looking for an analog theory to M-Theory for describing non-perturbative compactifications of Type IIB theory (for a review see [18,15]). Usually in perturbative compactifications the parameter $\lambda = a + iexp(-\Phi/2)$ is taken to be

constant. F-theory generalizes this fact by considering variable λ. Thus F-theory is defined as a twelve-dimensional theory whose compactification on the elliptic fibration $T^2 - \mathcal{M} \to D$, gives the Type IIB theory compactified on D (for a suitable space D) with the identification of $\lambda(\vec{z})$ with the modulus $\tau(\vec{z})$ of the torus T^2. These compactifications can be related to the M-theory compactifications through the known S mapping $\mathcal{S} : IIA \to M/\mathbf{S}^1$ and the $\mathcal{T}$ map between Type IIA and IIB theories. This gives

$$F/\mathcal{M} \times \mathbf{S}^1 \Longleftrightarrow M/\mathcal{M}. \tag{30}$$

Thus the spectrum of massless states of F-theory compactifications can be described in terms of M-theory. Other interesting F-theory compactifications are the Calabi-Yau compactifications

$$F/CY \Leftrightarrow Het/K3. \tag{31}$$

4.8 Gravitational Duality

As a matter of fact, string theory constitutes nowadays the only consistent and phenomenologically acceptable way to quantize gravity. It contains in its low energy limit Einstein gravity. Thus, a legitimate question is the one of which is the 'dual' theory of gravity or, more precisely, how gravity behaves under duality transformations.

Gravitational analogs of non-perturbative gauge theories were studied several years ago, particularly in the context of gravitational Bogomolny bound. As recently was shown [23], there are additional non-standard p-branes in $D = 10$ type II superstring theory and $D = 11$ M-theory, and which are required by U-duality. These branes were termed 'gravitational branes' ('G-branes'), because they carry global charges which correspond to the ADM momentum P_M and to its 'dual', a $(D-5)$-form $K_{M_1 \cdots M_{D-5}}$, which is related to the NUT charge. These charges are 'dual' in the same sense that the electric and magnetic charges are dual in Maxwell theory, but they appear in the purely gravitational sector of the theory. Last year, Hull has shown in [23] that these global charges P and K arise as central charges of

the supersymmetric algebra of type II superstring theory and M-theory. Thus the complete spectrum of BPS states should include the gravitational sector.

Finally, a different approach to the 'gravitational duality' was worked out by using some techniques of strong-weak coupling duality for non-supersymmetric Yang-Mills theories were applied to the MacDowell-Mansouri dynamical gravity (for a review see [24]). One would suspect that both approaches might be related in some sense. One could expect that the gauge theory of gravity would be realized as the effective low energy theory on the 'G-branes'.

V. NON-BPS BRANES AND K-THEORY

5.1 Non-BPS Branes

The notion of D-branes as BPS states implies the existence of certain supersymmetric theory on the world-volume of the D-brane. However it is extremaly relevant the consideration of non-supersymmetric theories (in order to describe our non-supersymmetric world) and here is where it is important the construction of brane configurations without remanent supersymmetry. A. Sen proposed the construction of such non-supersymmetric configurations by considering pairs of D-branes and anti-D-branes (for a nice review see [25] see also [26,27]). These configurations break all supersymmetry and the spectrum on the world-volume has a tachyon which cannot be cancelled by **GSO** projection. The presence of this tachyon leads to unstable brane configuration and the configuration decay into an stable BPS configuration . The classification of these stable D-branes was given by Witten in terms of topological K-Theory in the beautiful seminal paper [28] (for a review of this exciting subject see [29]).

In order to fix some notation let X be the ten-dimensional spacetime manifold and let W be a $(p + 1)$-dimensional submanifold of X. Branes or antibranes or both together can be wrapped on W. When configurations of N coincident branes or antibranes only are wrapped on W, the world-volume spectra on W consists of a vector multiplet and scalars

in some representation of the gauge group. These configurations can be described through Chan-Paton bundles which are U(N) gauge bundles E over W for Type II superstring theory and by SO(N) or Sp(N) bundles in Type I theory. Gauge fields from the vector multiplet define a U(N) gauge connection for Type II theory (or SO(N) or Sp(N) gauge connection for Type I theory) on the (corresponding) Chan-Paton bundle. **GSO** projection cancels the usual tachyonic degrees of freedom. Something similar occurs for the anti-brane sector.

The description of coincident N_1 coincident p-branes and N_2 p-anti-branes wrapped on W leads to the consideration pairs of gauge bundles (E, F) (over W) with their respective gauge connections A and A'. In the mixed configurations **GSO** projection fails to cancel the tachyon. Thus the system is unstable and may flow toward the annihilation of the brane-antibrane pairs with RR charge for these brane configurations being conserved in the process.

On the open string sector Chan-Paton factors are 2×2 matrices constructed from the possible open strings stretched among the different types of branes. Brane-brane and antibrane-antibrane sectors correspond to the diagonal elements of this matrix. Off-diagonal elements correspond with the Chan-Paton labels of an oriented open string starting at a brane and ending at an antibrane and the other one to be the open string with opposite orientation.

The physical mechanism of brane-antibrane creation or annihilation without violation of conservation of the total RR charge, leads to consider physically equivalent configurations of N_1 branes and N_2 antibranes and the same configuration but with additional created or annihilated brane-antibrane pairs.

5.2 D-branes and K-Theory

The relevant mathematical structure describing the brane-antibrane pairs in general type I and II superstring theories is as follows:

1. G_1 and G_2 gauge connections A and A' on the Chan-Paton bundles E and F over W, respectively. Bundles E and F corresponding to branes and antibranes are topologi-

cally equivalent. The groups G_1 and G_2 are restricted to be unitary groups for Type II theories and symplectic or orthogonal groups for Type I theories.

2. Tachyon field T can be seen as a section of the tensor product of bundles $E \otimes F^*$ and its conjugate $\bar{T}$ as a section of $E^* \otimes F$ (where $*$ denotes the dual of the corresponding bundle.)

3. Brane-antibrane configurations are described by pairs of gauge bundles (E, F).

4. The physical mechanism of brane-antibrane creation or annihilation of a set of m 9-branes and 9-antibranes is described by the same $\mathrm{U}(m)$ (for Type II theories) or $\mathrm{SO}(m)$ (for type I theories) gauge bundle H. This mechanism is described by the identification of pairs of gauge bundles (E, F) and $(E \oplus H, F \oplus H)$. Actually instead of pairs of gauge bundles one should consider classes of pairs of gauge bundles $[(E, F)] = [E] - [F]$ identified as above. Thus the brane-antibrane pairs really determine an element of the K-theory group $\mathrm{K}(X)$ of gauge bundles over X and the brane-antibrane creation or annihilation of pairs is underlying the K-theory concept of *stable equivalence* of bundles. For 9-branes, the embedded submanifold W coincides with X and the thus brane charges take values in K-theory group of X.

Consistency conditions for 9-branes ($p = 9$) in Type IIB superstring theory such as tadpole cancellation implies the equality of the ranks of the structure groups of the bundles E and F. Thus $rk(G_1) = rk(G_2)$. The 'virtual dimension' d of an element (E, F) is defined by $d = rk(G_1) - rk(G_2)$. Thus tadpole cancellation leads to a description of the theory in terms of pairs of bundles with virtual dimension vanishing, $d = 0$. This is precisely the definition of reduced K-theory $\tilde{\mathrm{K}}(X)$. Thus consistency conditions implies to project the description to reduced K-theory.

In Type I string theory $9 - \bar{9}$ pairs are described by a class of pairs (E, F) of $\mathrm{SO}(N_1)$ and $\mathrm{SO}(N_2)$ gauge bundles over X. Creation-annihilation is now described through the

SO(k) bundle H over X. In Type I theories tadpole cancellation condition is $N_1 - N_2 = 32$. In this case equivalence class of pair bundles (E, F) determines an element in the *real* K-theory group KO(X). Tadpole cancellation $N_1 - N_2 = 32$, newly turns out into reduced real K-theory group $\tilde{\text{KO}}(X)$.

Type IIA theory involves more subtle. It was argued by Witten in [28] that configurations of brane-antibrane pairs are classified by the K-theory group of spacetime with an additional circle space $\mathbf{S}^1 \times X$. K-theory group for type IIA configurations is K($\mathbf{S}^1 \times X$).

5.3 Ramond-Ramond Fields and K-Theory

Ramond-Ramond charges are classified according to the K-theory groups. In this subsection we will review that the proper RR fields follows a similar classification. For details see the recent papers by Witten [30] and by Moore and Witten [31].

It can be showed that RR fields do not satisfy the Dirac quantization condition. Thus for example,

$$\int_{W_p} \frac{G_p}{2\pi} \notin \mathbf{Z}. \tag{32}$$

The reason of this is the presence of chiral fermions on the brane. The phase of the fermions contribute with a gravitational term $\lambda = \int_W \frac{1}{16\pi^2} tr(R \wedge R)$. This gives a correction to the Dirac quantization. In trying to extended it for the all RR fields G_p in string theory it is necessary introduce new ideas as the notion of quantum self-duality and K-theory. Thus RR fields should be generalized in the context of K-Theory and we will see that in fact, they find an appropriate description within this context. Similar as the RR charge, the RR fields find a natural classification in terms of K-theory.

For self-dual RR fields it is a very difficult to find the quantum partition function. For the scalar field in two-dimensions it can be obtained by summing over only one of the periods of the 2-torus. It is not possible to sum simultaneously over both periods. This description can be generalized to any higher degree p-forms G_p. It can be done by defining a function $\Omega(x)$ for x in the lattice $\{H^1(\Sigma, \mathbf{Z})\}$ of periods such that

$$\Omega(x + y) = \Omega(x)\Omega(y)(-1)^{(x,y)}, \tag{33}$$

where $(x, y) \equiv \int x \cup y = \int x \wedge y$.

The partition function can be constructed easily from these data. One first has to identify the corresponding period lattice λ. After that, find the Ω function as a **Z**-valued function on Λ such that it satisfies Eq. (33). Finally one has to construct the partition function.

Period Lattice for Ramond-Ramond Fields

Let X be the spacetime manifold. One could suppose that that period lattice are: $\oplus_{p\ even} H^p(X; \mathbf{Z})$ for Type IIA theory and $\oplus_{p\ odd} H^p(X; \mathbf{Z})$ for Type IIB theory. However this are not the right choice since the RR charges and fields actually take values in K-Theory, just as has been described in the last subsection. Thus, one can see that the period lattice for Type IIA theory is $K(X)$ and for Type IIB it is $K^1(X)$. This is more obvious from the anomalous brane couplings. If $X = \mathbf{R} \times Y$ we have

$$\frac{dG}{2\pi} = \delta(Y)\sqrt{\hat{A}(Y)}ch(TX). \tag{34}$$

Hence the period lattice is constructed from $\oplus_{p\ even} H^p(X; \mathbf{R})$ generated by $\sqrt{\hat{A}}ch(TX)$ for $x = (E, F) \in K(X)$. Still it is necessary to quantize the lattice by finding the Ω function and its corresponding quantum partition function.

The Ω Function

In K-Theory there exist a natural definition of the Ω function given by the index theory

$$(x, y) = \text{Index of Dirac Operator on X with values in x} \otimes \bar{y} = \int_X \hat{A}(X)ch(x)ch(\bar{y}). \tag{35}$$

Thus the Ω function can be defined as

$$\Omega(x) = (-1)^{j(x)}, \tag{36}$$

where $j(x)$ is given by the mod 2 index of the Dirac operator with values in the real bundle $x \otimes \bar{x}$. It can be shown that this definition of $\Omega(x)$ satisfies the relation (33). From this

one can construct a quantum partition function which is compatible with (i) T-duality, (ii) Self-duality of RR fields, (iii) the interpretation of RR fields in K-theory and (iv) description of the brane anomalies.

VI. STRING THEORY AND NONCOMMUTATIVE GAUGE THEORY

6.1 Noncommutative D-branes From String Interactions

Finally in this section we describe briefly some new developments on the relation between string theory and Connes's noncommutative Yang-Mills theory (for a survey on noncommutative geometry see the classic Connes book [32]). We do not pretend to be exhaustive but only to remark the key points of the recent exciting developments [33–35] (for a nice review see [36]).

The roughly idea consists from the description of a string propagating in a flat background (spacetime) of metric g_{ij} and a NS constant B-field B_{ij}. The action is given by

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \int_D d^2\sigma \left(g_{ij}\partial_a X^i \partial^a X^j - 2\pi i\alpha' B_{ij}\varepsilon^{ab}\partial_a X^i \partial_b X^j \right) \tag{37}$$

where D is the disc. Or equivalently

$$\mathcal{L} = \frac{1}{4\pi\alpha'} \int_D d^2\sigma \; g_{ij}\partial_a X^i \partial^a X^j - \frac{i}{2} \int_{\partial D} d\tau B_{ij} X^i \partial_\tau X^j$$

Equations of motion from this action are subjected to the boundary condition

$$g_{ij}\partial_n X^j + 2\pi i\alpha' B_{ij}\partial_t X^j|_{\partial\Sigma} = 0. \tag{38}$$

The propagator of open string vertex operators is given by

$$\langle X^i(\tau)X^j(\tau')\rangle = -\alpha' G^{ij}\log(\tau-\tau')^2 + \frac{i}{2}\Theta^{ij}\varepsilon(\tau-\tau') \tag{39}$$

where

$$G^{ij} = \left(\frac{1}{g+2\pi\alpha'B}\right)^{ij}_S, \qquad \Theta^{ij} = 2\pi\alpha' \left(\frac{1}{g+2\pi\alpha'B}\right)^{ij}_A. \tag{40}$$

Here S and A stands for the symmetric and antisymmetric part of the involved matrix, and the logarithmic term determines the anomalous dimensions as usual. Thus G_{ij} is the effective metric seen by the open strings. While, as was suggested by Schomerus [34], the antisymmetric part Θ^{ij} determines the *noncommutativity*.

The product of tachyon vertex operators $exp(ip \cdot X)$ and $exp(iq \cdot X)$ for $\tau > \tau'$ in the short distance singularity is written as

$$exp\Big(ip \cdot X\Big)(\tau)exp\Big(iq \cdot X\Big)(\tau') \sim (\tau - \tau')^{2\alpha' G^{ij}p_i q_j} \cdot exp\Big(-\frac{1}{2}\Theta^{ij}p_i q_j\Big)exp\Big(i(p+q) \cdot X\Big)(\tau') + \dots$$

$$(41)$$

or

$$exp\Big(ip \cdot X\Big) * exp\Big(iq \cdot X\Big) \sim exp\Big(ip \cdot X\Big) * exp\Big(iq \cdot X\Big) \equiv exp\Big(\frac{i}{2}\Theta^{ij}p_i q_j\Big)exp\Big(i(p+q) \cdot X\Big)$$

$$(42)$$

where $*$ is defined for any smooth functions F and G over X and it is given by

$$F * G = exp\Big(\frac{i\hbar}{2}\Theta^{ij}\frac{\partial}{\partial u_i}\frac{\partial}{\partial v_j}\Big)F(x+u)G(x+y).$$

$$(43)$$

Here the operation $*$ is associative $F * (G * H) = (F * G) * H$ and noncommutative $F * G \neq G * F$. The above product can be written as $F * G = FG + i\{F, G\} + \dots$ where $\{F, G\}$ is the Poisson bracket given by $\Theta^{ij}\partial_i F \partial_j G$. Θ is determined in terms of B. Its give an associative and noncommutative algebra. In the limit $\alpha' \to 0$ (ignoring the anomalous dimensions of open string sector) the product of vertex operators turns out to be the Moyal product of functions on the spacetime X.

Now one can consider scattering amplitude (parametrized by G and Θ) of k gauge bosons of momenta p_i, polarizations ε_i and Chan-Paton wave functions λ_i, $\imath = 1, \dots, k$

$$A(\lambda_i, \varepsilon_i, p_i)_{G,\Theta} = Tr\Big(\lambda_1\lambda_2 \dots \lambda_k\Big)\int d\tau'_i \langle \prod_{i=1}^{k} \varepsilon_i \cdot \frac{dX}{d\tau}exp\Big(ip_i \cdot X\Big)(\tau'_i)\rangle_{G,\Theta}.$$

$$(44)$$

The Θ dependence come from the factor $exp\Big(-\frac{i}{2}\sum_{s>r} p_i^{(s)}p_j^{(r)}\Theta^{ij}\Big)$. Thus amplitude factorizes as $A(\lambda_i, \varepsilon_i, p_i)_{G,\Theta=0} \cdot exp\Big(-\frac{i}{2}\sum_{s>r} p_i^{(s)}\Theta^{ij}p_j^{(r)}\varepsilon(\tau_r - \tau_s)\Big)$ which depends only on the cycle ordering of the points $\tau_1, \dots, \tau_k$ on the boundary of the disc ∂D.

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For $B = 0$ the effective action is obtained under the assumption that the divergences are regularized through the Pauli-Villars procedure and it is given by

$$S_G = \frac{1}{g_{st}} \int d^n x \sqrt{G} \Big(Tr F_{ij} F^{ij} + \alpha' \text{ corrections} \Big). \tag{45}$$

The important case of the effective theory when $\Theta \neq 0$ is incorporated through the phase factor and thus one have to replace the ordinary multiplication of wave functions by the $*$ product (effective action is computed by using point splitting regularization)

$$\hat{S}_G = \frac{1}{g_{st}} \int d^n x \sqrt{G} G^{ii'} G^{jj'} \Big(Tr \hat{F}_{ii'} * \hat{F}^{jj'} + \alpha' \text{ corrections} \Big), \tag{46}$$

where $\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_j \hat{A}_i - i\{\hat{A}_i, \hat{A}_j\}_M$ is the noncommutative field strength. Here $\{F, G\}_M \equiv F * G - G * F$. Thus we get a noncommutative Yang-Mills theory as the Θ (or B) dependence of the effective action to all orders in α'. Gauge field transformation $(\hat{\lambda} * \hat{A})_{ij} = \hat{\lambda}_{ik} * A_j^k$ and $\delta \hat{A}_i = \partial_i \hat{\lambda} + i\hat{\lambda} * \hat{A}_i - i\hat{A}_i * \hat{\lambda}$.

For the low varying fields the effective action is given by the Born-Infeld-Dirac action

$$S = \frac{1}{g_{st}(\alpha')^2} \int d^n x \sqrt{\det(g + \alpha'(F + B))}. \tag{47}$$

The same effective action is described by noncommutative Yang-Mills theory but also by the standard Yang-Mills theory. They differ only in the regularization prescription. For the standard commutative case it is the Pauli-Villars one, while for the noncommutative case it is the point splitting prescription. The two frameworks are equivalent and thus there is a redefinition of the variable fileds and it can be seen 'as a transformation connecting standard and noncommutative descriptions. The change of variables known as the *Seiberg-Witten map* is as follows

$$\hat{A}_i = A_i - \frac{1}{4}\Theta^{kl}\{A_k, \partial_l A_i + F_{li}\} + O(\Theta^2)$$

$$\hat{\lambda} = \lambda + \frac{1}{4}\Theta^{kl}\{\partial_l \lambda, +A_j\} + O(\Theta^2). \tag{48}$$

6.2 String Theory and Deformation Quantization

Very recently a renewed deal of excitation has been taken place in deformation quantization theory [37], since the Kontsevich's seminal paper [38]. In this paper Kontsevich proved by construction the existence of a star-product for any finite dimensional Poisson manifold. His construction is based on his more general statement known as the "formality conjecture". The existence of such a star-product determines the existence of a deformation quantization for any Poisson manifold. Kontsevich's proof was strongly motivated by some perturbative issues of string theory and topological gravity in two-dimensions, such as, matrix models, the triangulation of the moduli space of Riemann surfaces and mirror symmetry.

One of the main lessons of the stringly [38] and D-brane [34] descriptions of Kontsevich's formula in that of the deformation quantization for any Poisson manifold requires necessarily of string theory. In addition this was confirmed in [35]. The deformation parameter of this quantization is precisely the string scale α' (or the string coupling constant) which in the limit $\alpha' \to 0$ it reproduces the field theory limit but in this limit the deformation quantization does not exist. The deformation arising precisely when $\alpha' \neq 0$ is an indication that deformation quantization is an stringly phenomenon. Actually it was already suspected since the origin of the formality conjecture where several mathematical ingredients of string theory were present.

String action in a background NS constant B field is

$$S = \frac{1}{4\pi\alpha'} \int_D d^2z \partial_a X^i \partial_a X^j G_{ij} + \frac{1}{4\pi\alpha'} \int_D dz d\bar{z} J^i(z) \bar{J}^j(\bar{z}) B_{ij}, \tag{49}$$

where $J^i(x) = 2i\partial X^i(z,\bar{z})$ and $\bar{J}^i(x) = 2i\bar{\partial} X^i(z,\bar{z})$.

Define the function

$$F(X(x)) = V[F](x) := \frac{1}{(2\pi)^{d/2}} \int d^d k \hat{F}(k) V_k(x) \tag{50}$$

where $V_k(x) =: exp(ik_i X^i(x)):$ is the vertex operator. OPE between $J's$ and $V's$ operators leads to $V[F](1)V[G](0) \sim V[FG](0) + \dots$ The introduction of a NS constant B field in the action 'deforms' the OPE leading to

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$$(V[F](1)V[G](0)^B \sim V[F*G](0) + \dots \tag{51}$$

where the $*$ product will be determined. It can be obtained by computing the N-point correlations functions for the complete action (49) (including the B-term)

$$\langle \Phi_1 \Phi_2 \dots \Phi_N \rangle^B = \frac{1}{Z} \langle \Phi_1 \Phi_2 \dots \Phi_N exp\Big(-\frac{1}{4\pi\alpha'} \int_{\mathcal{H}} dz d\bar{z} J^i(z) \bar{J}^j(\bar{z}) B_{ij} \Big) \rangle^B$$

$$= \frac{1}{Z} \sum_{n=0}^{\infty} (-\frac{1}{4\pi\alpha'}) \frac{1}{n!} \int_{\mathcal{H}_n^\varepsilon} d^d z d^d \bar{z} \langle \Phi_1 \Phi_2 \dots \Phi_N \prod_{a=1}^{n} B_{i_a j_a} J^{i_a}(z_a) \bar{J}^{i_a}(\bar{z}_a) \rangle \tag{52}$$

where $Z := \langle exp\Big(-\frac{1}{4\pi\alpha'} \int_{\mathcal{H}} dz d\bar{z} J^i(z) \bar{J}^j(\bar{z}) B_{ij} \Big) \rangle^B$ and $\mathcal{H}_n^\varepsilon := \{(z_1, z_2, \dots, z_n)| Im(z_a) > \varepsilon, |z_a - z_b| > \varepsilon$ for $a \neq 0\}$. We choice $\Phi_1 = V[F](1)$ and $\Phi_2[G](0)$. Now using the usual OPE of the $J's$ and $V'_k s$ operators and substituting all this at Eq. (52) we get deduce the explicit form fo the $*$ product and it is given by

$$F * G = \sum_n (4\pi\alpha')^n W_n B_n(F,G), \tag{53}$$

where W_n are the weight function

$$W_n := \frac{1}{(2\pi)^{2n}} \frac{1}{n!} \int d^n z d^n \bar{z} \prod_{a=1}^{n} \Big(\frac{1}{z_a - 1} \frac{1}{\bar{z}} - \frac{1}{\bar{z} - 1} \frac{1}{z_a} \Big) \tag{54}$$

and $B_n(F,G)$ are bi-differential operators

$$B_n(F,G) := \sum \Theta^{i_1 j_1} \Theta^{i_2 j_2} \dots \Theta^{i_n j_n} \partial_{i_1} \partial_{i_2} \dots \partial_{i_n} F \ \partial_{j_1} \partial_{j_2} \dots \partial_{j_n} G, \tag{55}$$

where Θ^{ij} is that given in Eq. (40).

This is precisely the formula given by Kontsevich in [38] for the $*$ product on any Poisson manifold. In this case the Poisson manifold is the spacetime X and Θ_{ij} is the Poisson-like structure.

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